Chapter 7

Quadratic assignment problems: Formulations and bounds

7.1 Introduction

Quadratic assignment problems (QAPs) belong to the most difficult combinatorial optimization problems. Because of their many real-world applications, many authors have investigated this problem class. For a monograph on QAPs, see the book by Çela [176]. A volume with selected papers on this topic was edited by Pardalos and Wolkowicz [536]. Some of the more recent surveys are Burkard [130], Pardalos, Rendl, and Wolkowicz [535], Burkard, Çela, Pardalos, and Pitsoulis [142], Rendl [574], and Loiola, Maia de Abreu, Boaventura-Netto, Hahn, and Querido [464]. The Quadratic Assignment Problem Library (QAPLIB), set up by Burkard, Karisch, and Rendl [151] and currently maintained by P. Hahn at http://www.seas.upenn.edu/qaplib/, contains not only many test examples with computational results, but also a detailed bibliography on this topic and a survey on the latest results.

7.1.1 Models and applications

The QAP was introduced by Koopmans and Beckmann [427] in 1957 as a mathematical model for the location of indivisible economical activities. We want to assign n facilities to n locations with the cost being proportional to the flow between the facilities multiplied by the distances between the locations plus, eventually, costs for placing the facilities at their respective locations. The objective is to allocate each facility to a location such that the total cost is minimized. We can model this assignment problem by means of three $n \times n$ matrices:

$A = (a_{ik})$, where a_{ik} is the flow from facility i to facility k;

$B = (b_{jl})$, where b_{jl} is the distance from location j to location l;

$C = (c_{ij})$, where c_{ij} is the cost of placing facility i at location j.

The QAP in Koopmans–Beckmann form can now be written as

$$\min_{\varphi \in \mathcal{S}_n} \left(\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i=1}^{n} c_{i\varphi(i)} \right), \tag{7.1}$$

where, as usual, $\mathcal{S}_n$ is the set of all permutations of the integers $1, 2, \ldots, n$. Each individual product $a_{ik} b_{\varphi(i)\varphi(k)}$ is the transportation cost caused by assigning facility i to location $\varphi(i)$ and facility k to location $\varphi(k)$. Thus each term $c_{i\varphi(i)} + \sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)}$ is the total cost given, for facility i, by the cost for installing it at location $\varphi(i)$, plus the transportation costs to all facilities k, if installed at locations $\varphi(1), \varphi(2), \ldots, \varphi(n)$.

An instance of the QAP with input matrices A, B, and C is denoted by $QAP(A, B, C)$. If there is no linear term (hence, no matrix C), we just write $QAP(A, B)$.

In many cases matrix B fulfills the triangle inequality $b_{jl} + b_{lr} \geq b_{jr}$ for all j, l, and r. In these cases $QAP(A, B)$ is called *metric QAP*.

A more general version of the QAP was considered by Lawler [445]. Lawler introduced a four-index cost array $D = (d_{ijkl})$ instead of the three matrices A, B, and C and obtained the general form of a QAP as

$$\min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{k=1}^{n} d_{i\varphi(i)k\varphi(k)}. \tag{7.2}$$

The relationship with the Koopmans–Beckmann problem is

$$\begin{aligned} d_{ijkl} &= a_{ik} b_{jl} && (i, j, k, l = 1, 2, \ldots, n;\ i \neq k \text{ or } j \neq l); \\ d_{ijij} &= a_{ii} b_{jj} + c_{ij} && (i, j = 1, 2, \ldots, n). \end{aligned}$$

Example 7.1. We consider a Koopmans–Beckmann problem $QAP(A, B, C)$ with $n = 3$ and input matrices

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 4 & 7 \\ 5 & 6 & 2 \end{pmatrix}, \quad \text{and } C = \begin{pmatrix} 9 & 7 & 9 \\ 6 & 5 & 7 \\ 8 & 9 & 8 \end{pmatrix}.$$

Given a permutation, say, $\varphi = (2, 1, 3)$, we can easily compute the corresponding objective function value by first permuting the rows and columns of B according to φ, as in

$$B_\varphi = (b_{\varphi(i)\varphi(k)}) = \begin{pmatrix} 4 & 1 & 7 \\ 3 & 2 & 6 \\ 6 & 5 & 2 \end{pmatrix},$$

and then deriving from (7.1)

$$z = (4 + 2 + 28) + (9 + 8 + 30) + (30 + 30 + 2) + (7 + 6 + 8) = 164.$$

In order to obtain the equivalent Lawler's form, we need to define the four-index matrix D. Let us represent it through n^2 square matrices D^{ij} of order n. Matrix D^{ij} is formed by the elements d_{ijkl} with fixed indices i and j and variable indices $k, l = 1, 2, \dots, n$, for example,

$$D^{11} = \begin{pmatrix} 11 & 3 & 6 \\ 4 & 6 & 12 \\ 8 & 12 & 24 \end{pmatrix}.$$

In order to compute the objective function value corresponding to the same permutation $\varphi = (2, 1, 3)$, we need the matrices D^{12}, D^{21}, and D^{33}:

$$D^{12} = \begin{pmatrix} 1 & 11 & 7 \\ 2 & 8 & 14 \\ 4 & 16 & 28 \end{pmatrix}, D^{21} = \begin{pmatrix} 6 & 9 & 18 \\ 14 & 12 & 24 \\ 10 & 15 & 30 \end{pmatrix}, D^{33} = \begin{pmatrix} 25 & 30 & 10 \\ 30 & 36 & 12 \\ 5 & 6 & 10 \end{pmatrix}.$$

Note that each matrix D^{ij} has the cost $c_{i\varphi(i)}$ added to the element stored in row i and column j. We obtain

$$z = (11 + 2 + 28) + (9 + 14 + 30) + (30 + 30 + 10) = 164. \quad \blacksquare$$

It is astonishing how many real-life applications can be modeled as QAPs. An early natural application in location theory was used by Dickey and Hopkins [233] in a *campus planning* model. The problem consists of planning the sites of n buildings on a campus, where b_{jl} is the distance from site j to site l and a_{ik} is the traffic intensity between building i and building k. The objective is to minimize the total weekly walking distance between the buildings. Another early application in this area was described by Steinberg [623] who minimized the number of connections in a backboard wiring problem, nowadays an outdated technology of historical interest only. Elshafei [251] used QAPs in hospital planning. Bos [109] described a related problem for forest parks.

In addition to facility location, QAPs appear in a variety of applications such as computer manufacturing, scheduling, process communications, and turbine balancing. In the field of ergonomics Pollatschek, Gershoni, and Radday [551] as well as Burkard and Offermann [153] showed that QAPs can be applied to *typewriter keyboard design*. The problem is to arrange the keys on a keyboard so as to minimize the time needed to write texts. Let the set of integers $N = \{1, 2, \dots, n\}$ denote the set of symbols to be arranged. Then a_{ik} denotes the frequency of the appearance of the ordered pair of symbols i and k. The entries of the distance matrix b_{jl} are the times needed to press the key in position l after pressing the key in position j. A permutation $\varphi \in \mathcal{S}_n$ describes an assignment of symbols to keys. An optimal solution φ^* for the QAP minimizes the average time for writing a text. A similar application related to ergonomic design is the development of control boards in order to minimize eye fatigue by McCormick [484].

The *turbine runner problem* was originally studied by Bolotnikov [106] and Stoyan, Sokolovskii, and Yakovlev [625]. The blades of a turbine, which due to manufacturing have slightly different masses, should be welded on the turbine runner such that the center of gravity coincides with the axis of the runner. Mosevich [499], Schlegel [598], and Laporte and Mercure [446] applied the QAP model to this problem. It has been shown that the minimization of the distance between the center of gravity and the axis of the runner is $\mathcal{NP}$-hard, whereas the maximization can be obtained in polynomial time (see Burkard, Çela, Rote, and Woeginger [143] and Section 8.4).