Chapter 10

Multi-index assignment problems

10.1 Introduction

Multi-index assignment problems were introduced by Pierskalla [549] in 1968 as a natural extension of linear assignment problems. For a long time only 3-index assignment problems have been considered, while in recent years problems with more than 3 indices have been investigated, mainly in the context of multi-target tracking and data association problems (see, e.g., Poore [552, 553] and Poore, Rijavec, Liggins, and Vannicola [556]).

In the case of 3-index assignment problems two models have been investigated: the axial 3-index assignment problem and the planar 3-index assignment problem. (These names have been introduced by Schell [597] in 1955.) In the next section we describe the axial 3-index assignment problem, which in many respects resembles the classical assignment problem, but turns out to be $\mathcal{NP}$-hard. Therefore, we describe lower bound computations, polyhedral results, efficiently solvable special cases, and asymptotic results. The planar 3-index assignment problem is treated in Section 10.3. It has not been as thoroughly investigated as the axial 3-index assignment problem and is much harder to solve. In the last section we outline results on general multi-index assignment problems.

For surveys on multi-index assignment problems we refer the reader to Burkard and Çela [138] and Spieksma [618].

10.2 Axial 3-index assignment problem

The *axial 3-index assignment problem* (axial 3AP) can be stated in the following way. Let n^3 cost coefficients c_{ijk} $(i, j, k = 1, 2, \ldots, n)$ be given. We ask for two permutations φ and ψ such that $\sum_{i=1}^{n} c_{i\varphi(i)\psi(i)}$ is a minimum, i.e.,

$$\min_{\varphi,\psi\in\mathcal{S}_n} \sum_{i=1}^{n} c_{i\varphi(i)\psi(i)}, \tag{10.1}$$

where $\mathcal{S}_n$ denotes the set of all permutations of the integers $\{1, 2, \ldots, n\}$. Since the two permutations which describe a feasible solution can be chosen arbitrarily, the axial 3AP has

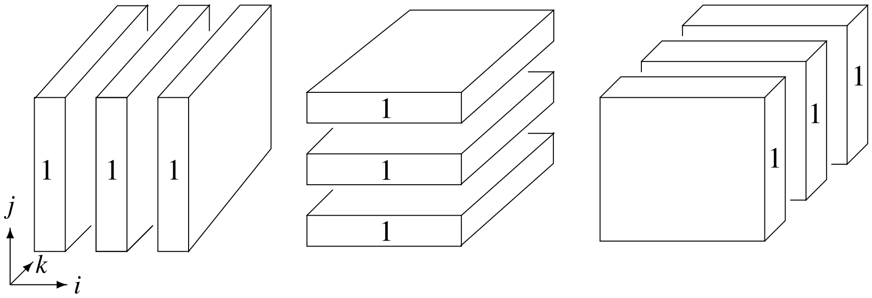

Figure 10.1. *Pictorial representation of the constraints of an axial 3AP.*

$(n!)^2$ feasible solutions. We can write this problem as an integer linear program:

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}x_{ijk} \tag{10.2}$$

$$\text{s.t.} \quad \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \qquad (i = 1, 2, \ldots, n), \tag{10.3}$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \qquad (j = 1, 2, \ldots, n), \tag{10.4}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} = 1 \qquad (k = 1, 2, \ldots, n), \tag{10.5}$$

$$x_{ijk} \in \{0, 1\} \qquad (i, j, k = 1, 2, \ldots, n). \tag{10.6}$$

Figure 10.1 gives a three-dimensional intuition of the constraints: a "1" on a face of the matrix means that exactly one 1 must be in that face.

As noted by Frieze [282], the axial 3AP can also be formulated as the following bilinear integer program with permutation matrices $Y = (y_{ij})$ and $Z = (z_{ik})$ (recall that $\mathbf{X}_n$ denotes the set of all $n \times n$ permutation matrices):

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}y_{ij}z_{ik} \\ \text{s.t.} \quad & Y \in \mathbf{X}_n, \\ & Z \in \mathbf{X}_n. \end{aligned}$$

Maximum weighted matchings in hypergraphs lead to another formulation of the axial 3AP. Consider the three-partite hypergraph $H = (V, E)$ whose $3n$ vertices correspond to the indices $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, n$ and whose edges are formed by triples (i, j, k). Let c_{ijk} be the cost of edge (i, j, k). A matching M in the hypergraph is a set of edges such that no two edges have a vertex in common. The axial 3AP asks for a maximum matching with minimum cost. This formulation is used in primal-dual bounding

It is interesting to note that the authors used, as a basis of their proofs, the so-called *index-trees* which were introduced by Pierskalla [549] in one of the first papers on multi-index assignment problems. In a recent paper, Krokhmal, Grundel, and Pardalos [435] developed lower and upper bounds on the expected optimal objective function value.

10.3 Planar 3-index assignment problem

The *planar 3-index assignment problem* (planar 3AP) can be formulated as follows. We say that n permutations $\varphi_1, \varphi_2, \dots, \varphi_n$ are *mutually distinct* if $\varphi_r(i) \neq \varphi_s(i)$ for any $i = 1, 2, \dots, n$ and $r \neq s$. Given n^3 cost coefficients c_{ijk} $(i, j, k = 1, 2, \dots, n)$, the problem is to find n mutually distinct permutations such that

$$\sum_{k=1}^{n} \sum_{i=1}^{n} c_{i\varphi_k(i)k} \tag{10.12}$$

is a minimum. The corresponding integer linear program is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk} x_{ijk} && && (10.13)\\
\text{s.t.} \quad & \sum_{k=1}^{n} x_{ijk} = 1 && (i, j = 1, 2, \dots, n), && (10.14)\\
& \sum_{i=1}^{n} x_{ijk} = 1 && (j, k = 1, 2, \dots, n), && (10.15)\\
& \sum_{j=1}^{n} x_{ijk} = 1 && (i, k = 1, 2, \dots, n), && (10.16)\\
& x_{ijk} \in \{0, 1\} && (i, j, k = 1, 2, \dots, n). && (10.17)
\end{aligned}$$

Frieze [285] showed that the problem is $\mathcal{NP}$-hard. Planar 3APs are closely related to Latin squares. A *Latin square* is an $n \times n$ array with entries l_{ij} taking the values from 1 to n. Every row and every column of a Latin square contains exactly one entry of value k $(1 \leq k \leq n)$. For example, a Latin square of size 4 may have the form

4	2	1	3
1	4	3	2
2	3	4	1
3	1	2	4

.

Every feasible solution of a planar 3AP can be represented as a Latin square. Let $L = (l_{ij})$ be a Latin square of size n. Then, for $i, j = 1, 2, \dots, n$, l_{ij} is the (unique) index value k such that $x_{ijk} = 1$ in a feasible solution of the planar 3AP. Thus the Latin square above corresponds to the following solution of a planar 3AP with $n = 4$:

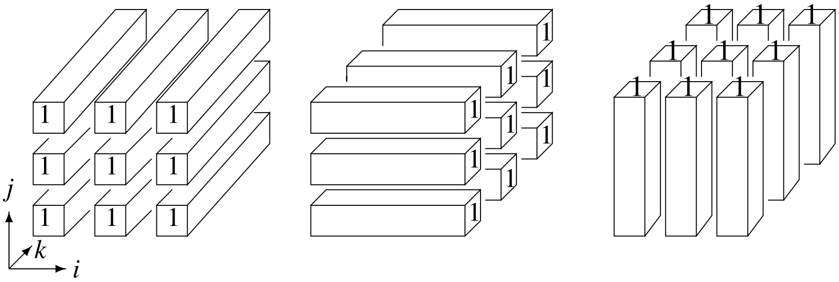

Figure 10.2. *Pictorial representation of the constraints of a planar 3AP.*

$$\begin{aligned}
x_{114} &= x_{122} = x_{131} = x_{143} = 1,\\
x_{211} &= x_{224} = x_{233} = x_{242} = 1,\\
x_{312} &= x_{323} = x_{334} = x_{341} = 1,\\
x_{413} &= x_{421} = x_{432} = x_{444} = 1.
\end{aligned}$$

This leads to the following geometric interpretation of the planar 3AP. Let us arrange the entries x_{ijk} in a cube. Then every plane in the cube, described either by i fixed, j fixed, or k fixed, respectively, must contain a (two-dimensional) assignment. Due to this interpretation, the number of feasible solutions of a planar 3AP of size n equals the number of Latin squares of order n, and hence increases very quickly. For example, the number of feasible solutions of a planar 3AP with $n = 9$ is $9! \cdot 8! \cdot 377,597,570,964,258,816 (\simeq 55 \cdot 10^{26})$ according to Bammel and Rothstein [66].

Figure 10.2 gives a three-dimensional intuition of the constraints: a "1" on a line of the matrix means that exactly one 1 must be in that line.

Constraints (10.14)–(10.16) show that the planar 3AP can be viewed as a matroid intersection problem on the ground set $E = \{(i, j, k) : i, j, k = 1, 2, \ldots, n\}$. Every block of constraints defines a partiton matroid on E. For fixed indices i and j, let $P^{ij} = \{(i, j, k) : k = 1, 2, \ldots, n\}$. Then $\mathcal{P}^{ij} = \{P^{ij} : i, j = 1, 2, \ldots, n\}$ yields a partition of the ground set E. In a similar way, we get two other partitions of E, namely, $\mathcal{P}^{ik} = \{P^{ik} : i, k = 1, 2, \ldots, n\}$ and $\mathcal{P}^{jk} = \{P^{jk} : j, k = 1, 2, \ldots, n\}$. A subset $F \subseteq E$ is a basis of $(E, \mathcal{P}^{ij})$ if, for all $i, j = 1, 2, \ldots, n$,

$$|F \cap P^{ij}| = 1.$$

In particular, $|F| = n^2$ holds. We get three partition matroids on the ground set E, namely, $(E, \mathcal{P}^{ij})$, $(E, \mathcal{P}^{ik})$, and $(E, \mathcal{P}^{jk})$. A common basis of these three matroids corresponds in a unique way to a feasible solution of a planar 3AP (Latin square) and vice versa. We ask for a basis in the intersection of the three partition matroids which has minimum cost. Similarly to the case of the axial 3AP, this model can be used for deriving lower bounds by subgradient methods related to a Lagrangean relaxation of the planar 3AP.

10.3.1 Applications and special cases

Planar 3APs play a crucial role in the context of timetabling problems. Consider an institute where n groups have to be taught by n lecturers in n different time slots. We assume that